The last supposition is absurd and hence we have again the triangle isosceles.

Case III. Suppose

$$BD = CE', \quad AC \times BC = AE' \times BE' - CE'^2, \quad AB \times BC = AD \times CD + BD^2,$$

$$(c + d)h = a'b' - f'^2, \tag{3}$$

$$(a' - b')h = cd + f'^2. \tag{4}$$

Also

$$AB : BC = AD : CD, \qquad AC : BC = E'A : E'B,$$

$$ch = (a' - b')d, \tag{5}$$

$$a'h = (c + d)b', \tag{6}$$

$$(c + d)h + (a' - b')h = a'b' + cd.$$

Introducing the values of $a'h$ and ch into (3) and (4) and reducing, we have

$$b'(c - h - a') = d(c - h - a'). \tag{7}$$

Hence

$$b' = d \text{ if } c \neq (a' + h);$$

$BDCE'$ is a parallelogram and the vertex A of the triangle ABC is at infinity.

If $c = a' + h$ we have from (7) the relation

$$\frac{b'}{d} = \frac{c - h - a'}{c - h - a'} = \frac{0}{0}$$

and we cannot infer $b' = d$.

If the triangle ABC is isosceles we have, dividing (5) by (6),

$$\frac{a' - b'}{c + d} = \frac{b'c}{a'd} = 1; \qquad \text{or} \qquad a' : b' = c : d.$$

The triangles ABD, $AE'C$ are similar and BD is parallel to $E'C$ and they cannot be equal. Therefore the triangle ABC is not isosceles.

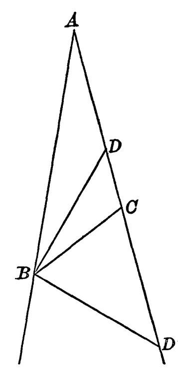

FIG. 2.

Case IV. Suppose $BD = BD'$.

This case can be very easily disposed of by noting that, since BD and BD' (Fig. 2) are at right angles to each other the triangle DBD' must be one half the square on BD with DD' as diagonal. Extend the diagonal in the direction $D'D$. Let any two lines from the point B, making equal angles with BD, meet $D'D$ in the points C and A. The triangle ABC evidently satisfies the conditions of the theorem and is not generally isosceles.

III. Relating to the Law of Cosines for a Polygon.

By F. M. MORGAN, Dartmouth College, Hanover, N. H.

The proof of the law of cosines for a plane triangle, as generally given in the texts on trigonometry, does not lend itself readily to a generalization that will

apply to a polygon either plane or skew. If, however, we make use of the theorems on projection in proving the law for a plane triangle, we can readily extend our proof to apply to the above configuration. Moreover, from the equations used in proving this general law, an identical relation among the angles of the polygon can be deduced.

Let us first derive the law of cosines for a plane triangle using the theory of projection. If we denote the sides of the triangle by the letters a_1, a_2, a_3, respectively, we have

$$\begin{aligned} a_1 &= a_2 \cos(a_1a_2) + a_3 \cos(a_1a_3), \\ a_2 &= a_1 \cos(a_2a_1) + a_3 \cos(a_2a_3), \\ a_3 &= a_1 \cos(a_3a_1) + a_2 \cos(a_3a_2). \end{aligned} \tag{1}$$

Let us now multiply these equations by a_1, $-a_2$, $-a_3$, respectively, and add, remembering that $\cos(a_ia_j) = \cos(a_ja_i)$. We then obtain

$$a_1^2 = a_2^2 + a_3^2 - 2a_2a_3 \cos(a_2a_3)$$

which is the well-known law of cosines.

If we eliminate a_1, a_2, a_3 from equations (1) we obtain

$$\begin{vmatrix} -1 & \cos(a_1a_2) & \cos(a_1a_3) \\ \cos(a_1a_2) & -1 & \cos(a_2a_3) \\ \cos(a_1a_3) & \cos(a_2a_3) & -1 \end{vmatrix} = 0, \tag{2}$$

which when simplified gives

$$\cos^2(a_1a_2) + \cos^2(a_1a_3) + \cos^2(a_2a_3) + 2\cos(a_1a_2)\cos(a_1a_3)\cos(a_2a_3) = 1$$

as an identical relation between the angles of the triangle.[1]

Let us now consider a general polygon whose sides we will denote by $a_1, a_2, \cdots, a_n$ respectively. Then

$$\begin{aligned} a_1 &= a_2 \cos(a_1a_2) + a_3 \cos(a_1a_3) + \cdots + a_n \cos(a_1a_n), \\ a_2 &= a_1 \cos(a_2a_1) + a_3 \cos(a_2a_3) + \cdots + a_n \cos(a_2a_n), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_n &= a_1 \cos(a_na_1) + a_2 \cos(a_na_2) + \cdots + a_{n-1} \cos(a_na_{n-1}). \end{aligned} \tag{3}$$

If we multiply these equations by a_1, $-a_2$, $-a_3$, $\cdots$, $-a_n$ respectively and add we obtain

$$a_1^2 = a_2^2 + a_3^2 + \cdots + a_n^2 - 2[a_2a_3 \cos(a_2a_3) + a_2a_4 \cos(a_2a_4) + \cdots + a_{n-1}a_n \cos(a_{n-1}a_n)],$$

which we will call the law of cosines for a polygon. Moreover it should be noted that the proof is perfectly general, *i. e.*, it applies to a polygon that is either concave or convex, plane or skew.

[1] Paterson, *Elementary Trigonometry*, Oxford, p. 168. Prove $\Sigma \cos^2 A = 1 - 2\Pi \cos A$.

If we eliminate the a's from equations (3) we obtain

$$\begin{vmatrix} -1 & \cos(a_1a_2) & \cos(a_1a_3) & \cdots & \cos(a_1a_n) \\ \cos(a_1a_2) & -1 & \cdots\cdots & \cdots & \cos(a_2a_n) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cos(a_1a_n) & \cdot & \cdot & \cdot & -1 \end{vmatrix} = 0 \qquad (4)$$

as an identical relation between the cosines of the angles. For a quadrilateral (4) becomes, if we denote $\cos(a_ia_j)$ by (i, j),

$$1 - \Sigma(ij)^2 + [(12)(34) - (13)(24) - (14)(23)]^2 - 2(12)[(13)(23) + (14)(24)] - 2(34)[(13)(14) + (23)(24)] = 0.$$

UNDERGRADUATE MATHEMATICS CLUBS.

Edited by R. C. Archibald, Brown University, Providence, R. I.

CLUB ACTIVITIES.

The Mathematics Club, University of Colorado, Boulder, Colo.

This club was organized in October, 1915, "to stimulate interest in mathematics among those who have had calculus." The total membership this year is 41 and the average attendance about 30. Professor George H. Light acts as chairman of the meetings and the following program for 1917–18 was arranged by him "with the assistance of club members," and issued in printed form.

November 20: "Non-Euclidean Geometry" by Leroy A. MacColl '19;
December 4: "Discovery of Logarithms" by Leona E. Vincent '19;
December 18: "Squaring the Hyperbola" by Ada G. Hall '18; "Probability in Arithmetic" by Henry A. Howell '18;
January 15: "Condition that $f(x, y, z)$ can be factored" by Agnes M. Wright '20;
February 5: "Applications for Vectors" by Claribell Kendall, instructor in mathematics;
February 19: "Nth Dimensions" by Lauren C. Hand '19;
March 5: "Relativity in Astronomy" by Edgar W. Wollard '20;
March 19: "American Mathematicians" by Dorothy Bair '20, and Alfreda Alenius '21;
April 2: "Proofs of Pythagoras's Theorem" by Lila Nelson '20; "Geometric Proof that $\sin 3A = 3 \sin A - 4 \sin^3 A$" by Oliver De Motte Sp.;
April 16: "Certain Definite Integrals" by Mildred McMillen '19;
May 7: "Curve Tracing" by Anthony J. Killgore '20;
May 21: "Famous Problems in Mathematics" by Gussie Wellman '21.